Telegrapher's equations with variable propagation speeds

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All derivations of the one-dimensional telegrapher's equation, based on the persistent random walk model, assume a constant speed of signal propagation. We generalize here the model to allow for a variable propagation speed and study several limiting cases in detail. We also show the connections of this model with anomalous diffusion behavior and with inertial dichotomous processes.

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The persistent random walk, hereafter referred to as PRW, was suggested first by Fürth [1] and Taylor [2] as a model allowing one to incorporate a simple analogy of momentum in the framework of diffusion theory. When one starts from a one-dimensional PRW on a lattice and passes to the continuum limit one finds that the probability density function for the displacement at time t satisfies a telegrapher's equation instead of the diffusion equation obtained from the ordinary random walk picture.

The telegrapher's equation is considered a simple generalization of the diffusion equation that overcomes conceptual problems related to the infinite speed of signal propagation. The telegrapher's equation possesses a finite propagation speed which is bounded by a constant. In this paper we will consider a nonuniform lattice and explore the consequences of having a nonconstant propagation speed.

Let us start with a one-dimensional PRW on a lattice. Each step length is Δx and has duration Δt. Let p be the probability that two successive steps are given in the same direction and define the probability of a reversal to be $q = 1 - p$. In the diffusion limit Δx and Δt go to zero in such a way that the ratio $\Delta x/\Delta t$ tends to a finite quantity c having the dimensions of velocity (the speed of signal propagation). As we have mentioned it is generally assumed that c is a constant which corresponds to a uniform lattice. This is not the case here, since we will assume that c is a function of x and t

$$\frac{\Delta x}{\Delta t} \longrightarrow c(x,t) \qquad (\Delta x, \Delta t \to 0). \tag{1}$$

Let $a(x,t)$ [$b(x,t)$] denote the probability density function for the position of the random walker at time t while moving in the positive [negative] x direction. It is easily shown from the set of difference equations satisfied by $a(x,t)$ and $b(x,t)$ in the lattice picture [3] that, if in the continuum limit we assume

$$p = 1 - \frac{\Delta t}{2T} + O((\Delta t)^2),$$

where T is a constant having dimensions of time, then the assumption (1) leads to the following set of partial differential equations:

$$\frac{\partial a}{\partial t} = -c(x,t)\frac{\partial a}{\partial x} + \frac{1}{2T}(b-a), \tag{2}$$

$$\frac{\partial b}{\partial t} = c(x,t)\frac{\partial b}{\partial x} + \frac{1}{2T}(a-b). \tag{3}$$

We will assume that Eqs. (2) and (3) are to be solved subject to the initial conditions

$$a(x,0) = b(x,0) = \frac{1}{2}\delta(x - x_0). \tag{4}$$

Equations (2) and (3) can be combined into a single second-order differential equation for the total probability density function

$$p(x,t) = a(x,t) + b(x,t) \tag{5}$$

of the random walker. This equation reads

$$\frac{\partial}{\partial t}\left[\frac{1}{c(x,t)}\frac{\partial p}{\partial t}\right] + \frac{1}{Tc(x,t)}\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left[c(x,t)\frac{\partial p}{\partial x}\right], \tag{6}$$

and the initial conditions are

$$p(x,0) = \delta(x - x_0), \tag{7}$$

$$\left.\frac{\partial p}{\partial t}\right|_{t=0} = 0. \tag{8}$$

When $c(x,t) = c$ is a constant velocity one recovers from Eq. (6) the ordinary telegrapher's equation

$$\frac{\partial^2 p}{\partial t^2} + \frac{1}{T}\frac{\partial p}{\partial t} = c^2\frac{\partial^2 p}{\partial x^2}. \tag{9}$$

There are two more cases in which Eq. (6) leads to a telegrapher's equation. In one of these cases c only depends on x:

$$c = c(x)$$

and Eq. (6) reduces to

$$\frac{\partial^2 p}{\partial t^2} + \frac{1}{T}\frac{\partial p}{\partial t} = c(x)\frac{\partial}{\partial x}\left[c(x)\frac{\partial p}{\partial x}\right]. \tag{10}$$

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This equation can be written in the form of the ordinary telegrapher's equation. In effect, if $c(x)$ is such that $1/c(x)$ is an integrable function over the real line, the new coordinate

$$y = \int_{x_0}^{x} \frac{dx'}{c(x')} \tag{11}$$

converts Eq. (10) into the following equation:

$$\frac{\partial^2 p}{\partial t^2} + \frac{1}{T}\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial y^2}. \tag{12}$$

Therefore the solution to Eq. (10) under the initial conditions (7) and (8) is [3,4]

$$p(x,t) = \frac{e^{-t/2T}}{4|c(x)|}\Bigg\{2\delta\left(t - \left|\int_{x_0}^{x}\frac{dx'}{c(x')}\right|\right) + \frac{1}{T}\left(I_0\left[\frac{\lambda(x,t)}{2T}\right] + \frac{t}{\lambda(x,t)}I_1\left[\frac{\lambda(x,t)}{2T}\right]\right) \times\Theta\left(t - \left|\int_{x_0}^{x}\frac{dx'}{c(x')}\right|\right)\Bigg\}, \tag{13}$$

where $\Theta(x)$ is the Heaviside step function, $I_{0,1}(x)$ are modified Bessel functions and

$$\lambda(x,t) \equiv \left[t^2 - \left(\int_{x_0}^{x}\frac{dx'}{c(x')}\right)^2\right]^{1/2}. \tag{14}$$

Another situation where it is still possible to get a telegrapher's equation is when c is only a function of t

$$c = c(t).$$

In this case the change of time scale

$$\tau = \int_0^t c(t')\, dt' \tag{15}$$

converts Eq. (6) into the following telegrapher's equation with time varying coefficients:

$$\frac{\partial^2 p}{\partial \tau^2} + \frac{\chi(\tau)}{T}\frac{\partial p}{\partial \tau} = \frac{\partial^2 p}{\partial x^2}, \tag{16}$$

where

$$\chi(\tau) \equiv \frac{1}{c(t(\tau))} \tag{17}$$

and the function $t(\tau)$ is implicitly defined by Eq. (15). It is very difficult to find an exact solution to Eq. (16) for arbitrary $\chi(\tau)$. Nevertheless, it is possible to find an asymptotic solution to Eq. (16) valid when T is small. To this end we first Fourier transform Eq. (16) with the result

$$\frac{\partial^2 \tilde{p}}{\partial \tau^2} + \frac{\chi(\tau)}{T}\frac{\partial \tilde{p}}{\partial \tau} + \omega^2 \tilde{p} = 0, \tag{18}$$

where

$$\tilde{p}(\omega,\tau) = \int_{-\infty}^{\infty} e^{i\omega x} p(x,t)dx.$$

We now write $T = \epsilon T'$ where $\epsilon > 0$ is a dimensionless small parameter and look for an asymptotic solution in the form of a WKB series

$$\tilde{p}(\omega,t) \sim \exp\left[\frac{1}{\epsilon}\sum_{n=0}^{\infty}\epsilon^n S_n(\omega,t)\right].$$

The substitution of this equation into Eq. (18) yields

$$\tilde{p}(\omega,\tau) \sim \exp\left\{-\epsilon T'\omega^2\int_0^{\tau}\frac{dz}{\chi(z)} + O(\epsilon^2)\right\}$$

and its Fourier inversion reads

$$p(x,\tau) \sim \frac{1}{\sqrt{2\pi T}\sigma(\tau)}\exp\left[-\frac{x^2}{4T\sigma^2(\tau)}\right], \tag{19}$$

where

$$\sigma^2(\tau) \equiv \int_0^{\tau}\frac{dz}{\chi(z)}. \tag{20}$$

We note that the same result could have been obtained in a more heuristic way. In effect, when T is small we may assume that the telegrapher's equation (16) is approximated by the diffusion equation

$$\frac{\chi(\tau)}{T}\frac{\partial p}{\partial \tau} = \frac{\partial^2 p}{\partial x^2}, \tag{21}$$

whose solution is precisely given by Eq. (19). In fact, the WKB procedure outlined above provides a proof of the soundness of the approximation leading to Eq. (21) which, in turn, is related to the central limit theorem.

Let us now write Eq. (19) in the original time scale t. From the definition of $\chi(\tau)$ given by Eq. (17) we see that the "variance" $\sigma^2(\tau)$ can be written as [cf. Eq. (20)]

$$\sigma^2(\tau) = \int_0^{\tau} c(t(z))dz,$$

where $t(z)$ is implicitly defined by

$$z = \int_0^{t(z)} c(t')dt'.$$

Combining these two equations we get

$$\sigma^2(\tau) = \tau = \int_0^t c(t')dt'.$$

Hence

$$p(x,t) \sim \frac{\left(\int_0^t c(t')dt'\right)^{-1/2}}{\sqrt{4\pi T}}\exp\left[-\frac{x^2}{4T\int_0^t c(t')dt'}\right]. \tag{22}$$

Although this approximation has been derived when T is small, it is also known that the same approximation works for any value of T provided that $t \gg T$ [6]. Therefore Eq. (22) is the asymptotic solution to Eq. (6) when $c = c(t)$ and $t \gg T$.

Let us now suppose that as $t \to \infty$

$$c(t) \sim t^{\alpha}, \tag{23}$$

then

$$\sigma^2(t) = \int_0^t c(t')dt' \sim \frac{t^{\alpha+1}}{\alpha+1}. \tag{24}$$

We observe that this kind of variance corresponds to anomalous diffusion behavior, where if $\alpha > 0$ we obtain superdiffusive behavior while $-1 < \alpha < 0$ corresponds to subdiffusion.

We note that subdiffusion has been associated with the persistent random walk with long waiting times [7,8] while a model for superdiffusion seems to be difficult to obtain [10] although it has been shown that free inertial dichotomous processes have a superdiffusive behavior characterized by [4,9]

$$\sigma^2(t) \sim t^3.$$

We see that the model outlined above shows that the persistent random walk can lead to anomalous diffusion behavior *allowing for both subdiffusion and superdiffusion.*

Another interesting feature of the time-varying telegrapher's equation (16) is its closed relationship with dichotomous inertial processes. We have recently found [5] that for inertial dichotomous processes of the form

$$\ddot{X} + \beta\dot{X} = F(t), \tag{25}$$

where $F(t)$ is dichotomous Markov noise alternately taking on values $\pm c$ with an exponential switch density $\psi(t) = (1/2T)\exp(-t/2T)$, the marginal density $p(x,t)$ of the displacement obeys the following telegrapher's equation:

$$\frac{\partial^2 p}{\partial t^2} + \left(\frac{1}{T} - \beta\,\frac{e^{-\beta t}}{1-e^{-\beta t}}\right)\frac{\partial p}{\partial t} = \frac{c^2}{\beta^2}(1-e^{-\beta t})^2\frac{\partial^2 p}{\partial x^2}. \tag{26}$$

For the undamped case, $\beta = 0$, this equation reads

$$\frac{\partial^2 p}{\partial t^2} + \left(\frac{1}{T} - \frac{1}{t}\right)\frac{\partial p}{\partial t} = c^2 t^2\frac{\partial^2 p}{\partial x^2}. \tag{27}$$

The similarity between these equations and Eq. (16) is better seen if we go back to the original time scale t. In this scale Eq. (16) reads

$$\frac{\partial^2 p}{\partial t^2} + \left[\frac{1}{T} - \frac{\dot{c}(t)}{c(t)}\right]\frac{\partial p}{\partial t} = c^2(t)\frac{\partial^2 p}{\partial x^2}. \tag{28}$$

We thus see from Eqs. (26)–(28) that the equation for the marginal density $p(x,t)$ of the inertial process (25) coincides with the equation for the probability density function of a PRW on a line with a variable speed of signal propagation bounded by

$$c(t) = \frac{c}{\beta}(1 - e^{-\beta t}) \tag{29}$$

in the case of damped motion, and

$$c(t) = ct \tag{30}$$

in the case of free motion.

We finally note that this analogy might open an alternative way to study the difficult problem of finding marginal probability density functions for the displacement of inertial processes. This alternative procedure would consist in looking for an equivalent persistent random walk with a variable speed of signal propagation conveniently chosen. This point is under investigation.

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